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Approximate analytical solutions of the Dirac equation with the Pöschl–Teller potential including the spin–orbit coupling term

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Abstract
By employing an exponential-type potential to replace the spin–orbit coupling term, we solve approximately the Dirac equation for the Pöschl–Teller potential with the pseudospin symmetry for the arbitrary spin–orbit quantum number κ. The bound-state energy eigenvalues and the associated two-component spinors of the Dirac particles are obtained approximately.

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1. Introduction

Over thirty years ago, the pseudospin symmetry concept in nuclear theory was introduced [1, 2] and it has been used to explain features of deformed nuclei [3] and superdeformation [4], and to establish an effective shell-model coupling scheme [5]. Within the framework of the relativistic mean field theory, Ginocchio [6] found that the pseudospin symmetry in nuclei occurs when an attractive scalar potential $S(r)$ and a repulsive vector potential $V(r)$ have the near equality of the magnitude, i.e., $S(r) \sim -V(r)$. Meng *et al* [7] showed that the pseudospin symmetry is exact under the condition, $\mathrm{d}(V(r) + S(r))/\mathrm{d}r = 0$, and the quality of the pseudopsin approximation in real nuclei is connected with the competition between the pseudo-centrifugal barrier and the pseudospin-orbital potential. When the sum potential between the vector potential and scalar potential is a constant, i.e., $V(r) + S(r) =$ constant, the pseudospin symmetry occurs in the Dirac equation [8]. In recent years, some authors have studied the pseudospin symmetry for some typical physical potentials, such as the Coulomb

potential [9], harmonic oscillator [10–12], Woods–Saxon potential [13], ring-shaped non-spherical harmonic oscillator [14], Morse potential [15–17], Eckart potential [18–20], three-parameter potential function as a diatomic molecule model [21], Pöschl–Teller potential [22] and angle-dependent potential [23]. Within the framework of the Dirac equation, Alhaidari [24–26] has investigated the relativistic extensions for some shape-invariant potentials, which include the Morse, Rosen–Morse, Eckart, Pöschl–Teller, Scarf and Hulthén potentials.

The Pöschl–Teller potential [27] is an important diatomic molecular potential model which has been widely applied in physics and chemical physics. This potential can be used as the electron–nucleus potential to investigate the strong-field ionization dynamics of a simplified one-dimensional model of a homonuclear molecular ion [28]. The Pöschl–Teller potential is a typical anharmonic potential, which can be used to study the out-of-plane bending vibrations [29]. The Pöschl–Teller potential is also included in the five-parameter exponential-type potential model as a special case [30, 31]. For a diatomic molecular potential model, we consider the reduced mass. If the nuclei have masses m_1 and m_2, the reduced mass is defined as $\mu = m_1 m_2/(m_1 + m_2)$ and in this point the diatomic molecular model can be included to the pseudospin symmetry concept. By solving the Dirac equation with mixed potentials in terms of the different methods, some authors have investigated the pseudospin symmetry for some diatomic molecular potential, such as the Eckart potential [18–20], three-parameter potential function as a diatomic molecule model [21] and Pöschl–Teller potential [22]. In [32], the authors analyzed the bound-state solutions of the s-wave Klein–Gordon equation with equal scalar and vector Pöschl–Teller potentials by using the supersymmetric quantum mechanics method and the shape invariance approach. In [33], the authors investigated the bound-state solutions of the s-wave Dirac equation with equally mixed Pöschl–Teller potentials in terms of the supersymmetric quantum mechanics approach and the function analysis method. Using the same methods, the pseudospin symmetry solutions of the Dirac equation with the Pöschl–Teller potential for the spin–orbit quantum number $\kappa = 1$ have also been investigated [22]. However, as far as we know, one has not reported the investigation of the pseudospin symmetry solutions of the Dirac equation with the Pöschl–Teller potential for the arbitrary spin–orbit quantum number κ.

Motivated by the success made by some authors in finding approximate analytical solutions of the Schrödinger equation with the centrifugal potential term for the Hulthén potential [34], generalized Hulthén potential [35], Manning–Rosen potential [36], Eckart potential [37] and Schiöberg diatomic molecule potential [38], in the present work, we solve approximately the Dirac equation with the Pöschl–Teller potential for the arbitrary spin–orbit quantum number κ. Under the condition of the exact pseudospin symmetry, some authors [19, 20] have investigated the pseudospin symmetry solutions of the Dirac equation with the Eckart potential for any spin–orbit quantum number κ by employing an exponential-type potential to replace the spin–orbit coupling term.

2. Bound-state solutions

The Dirac equation for a fermion with mass M in a scalar potential $S(r)$ and a vector potential $V(r)$ can be written as ($\hbar = c = 1$)

$$\{\alpha \bullet \mathbf{p} + \beta[M + S(r)]\}\Psi(r) = [E - V(r)]\Psi(r), \tag{1}$$

where E is the relativistic energy of the system, $\mathbf{p}$ is the three-dimensional momentum operator, $\mathbf{p} = -\mathrm{i}\nabla$, α and β are the 4×4 Dirac matrices which, in the usual representation, are given

by the following forms, respectively,

$$\alpha = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{2}$$

where I is the 2×2 unit matrix, the subscript i can take the values of 1, 2 and 3, σ_1, σ_2 and σ_3 the three 2×2 Pauli matrices, i.e.,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

For a Dirac particle in a spherically symmetric potential field, the total angular momentum operator $\mathbf{J}$ and spin–orbit matrix operator $K = -\beta(\sigma \cdot \mathbf{L} + 1)$ commute with the Dirac Hamiltonian, where $\mathbf{L}$ is the orbital angular momentum. The eigenvalues of K are $\kappa = \pm(j + 1/2)$, where $\kappa = -(j + 1/2) < 0$ is for the aligned spin $j = l + 1/2$ ($s_{1/2}$, $p_{3/2}$, etc), and $\kappa = (j + 1/2) > 0$ is for the unaligned spin $j = l - 1/2$ ($p_{1/2}$, $d_{3/2}$, etc). The complete set of the conservative quantities can be taken as (H, K, J^2, J_z), the spinor wavefunctions can be classified according to their angular momentum number j, spin–orbit quantum number κ and the radial quantum number n. The pseudospin symmetry refers to a quasi-degeneracy of the single-nucleon doublets and can be characterized with the non-relativistic quantum numbers $(n, l, j = l + 1/2)$ and $(n - 1, l + 2, j = l + 3/2)$, where n, l and j are the single-nucleon radial, orbital and total angular momentum quantum numbers, respectively. Using a pseudo-orbital angular momentum $\tilde{l} = l + 1$ and a pseudospin angular momentum $\tilde{s} = 1/2$, one can express the total angular momentum as $j = \tilde{l} + \tilde{s}$. For example, $(3s_{1/2}, 2d_{3/2})$ can be denoted with pseudospin doublets $(2\tilde{p}_{1/2}, 2\tilde{p}_{3/2})$, where $j = \tilde{l} \pm \tilde{s}$ for the two states in the doublet. The spherically symmetric Dirac spinor eigenfunction can be written as follows:

$$\Psi_{n\kappa} = \frac{1}{r} \begin{bmatrix} F_{n\kappa}(r)\, Y^{l}_{jm}(\theta, \phi) \\ i G_{n\kappa}(r)\, Y^{\tilde{l}}_{jm}(\theta, \phi) \end{bmatrix}, \tag{4}$$

where the upper and lower components $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are real square-integral functions, $Y^{l}_{jm}(\theta, \phi)$ and $Y^{\tilde{l}}_{jm}(\theta, \phi)$ are the spherical harmonic functions, and m is the projection of the total angular momentum on the third axis. Substituting equation (4) into equation (3) leads us to obtain two coupled differential equations for the upper and lower spinor components $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ as follows, respectively,

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right) F_{n\kappa}(r) = [M + E_{n\kappa} - \Delta(r)] G_{n\kappa}(r), \tag{5a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right) G_{n\kappa}(r) = [M - E_{n\kappa} + \Sigma(r)] F_{n\kappa}(r), \tag{5b}$$

where $\Delta(r)$ and $\Sigma(r)$ denote the difference potential and sum potential between the vector potential and scalar potential, respectively, i.e., $\Delta(r) = V(r) - S(r)$, $\Sigma(r) = V(r) + S(r)$. By eliminating $G_{n\kappa}(r)$ in equation (5*a*) and $F_{n\kappa}(r)$ in equation (5*b*), we can immediately obtain two second-order differential equations for the upper and lower components as follows,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa + 1)}{r^2} - (M + E_{n\kappa} - \Delta)(M - E_{n\kappa} + \Sigma) + \frac{\frac{\mathrm{d}\Delta}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right)}{M + E_{n\kappa} - \Delta}\right) F_{n\kappa}(r) = 0, \tag{6a}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa - 1)}{r^2} - (M + E_{n\kappa} - \Delta)(M - E_{n\kappa} + \Sigma) - \frac{\frac{\mathrm{d}\Sigma}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right)}{M - E_{n\kappa} + \Sigma}\right) G_{n\kappa}(r) = 0. \tag{6b}$$

Under the condition of exact pseudospin symmetry, i.e., $\frac{d\Sigma}{dr} = 0$ or $\Sigma = C =$ constant, equation (6*b*) turns into the following form:

$$\left(-\frac{d^2}{dr^2} + \frac{\kappa(\kappa-1)}{r^2} - (M - E_{n\kappa} + C)\Delta(r)\right) G_{n\kappa}(r) = \left(E_{n\kappa}^2 - M^2 - C(M + E_{n\kappa})\right) G_{n\kappa}(r). \tag{7}$$

Equation (7) shows that the energy eigenvalues, $E_{n\kappa}$, depend only on n and κ, i.e., $E_{n\kappa} = E(n, \kappa(\kappa-1))$. In view of the relation $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$, the energy eigenvalues, $E_{n\kappa}$, also depend only on n and $\tilde{l}$, i.e., $E_{n\kappa} = E(n, \tilde{l}(\tilde{l}+1))$. For $\tilde{l} \neq 0$, the states with $j = \tilde{l} \pm 1/2$ are degenerate. This is the pseudospin symmetry. In equation (7), we set the difference potential $\Delta(r)$ as the Pöschl–Teller potential [27] given by

$$\Delta(r) = -\frac{A(A+\alpha)}{\cosh^2 \alpha r} + \frac{B(B-\alpha)}{\sinh^2 \alpha r}, \tag{8}$$

where the parameter α is related to the range of the potential. Considering that the potential (8) remains unchanged under the transformations of $A \to -A - \alpha$ and $B \to -B + \alpha$, we may only discuss the case of $B < A$. Substituting equation (8) into equation (7) yields the following second-order Schrödiner-like equation:

$$\left(-\frac{d^2}{dr^2} - 4(M - E_{n\kappa} + C)e^{-2\alpha r}\left(-\frac{A(A+\alpha)}{(1+e^{-2\alpha r})^2} + \frac{B(B-\alpha)}{(1-e^{-2\alpha r})^2}\right) + \frac{\kappa(\kappa-1)}{r^2}\right) G_{n\kappa}(r)$$
$$= \left(E_{n\kappa}^2 - M^2 - C(M + E_{n\kappa})\right) G_{n\kappa}(r). \tag{9}$$

This equation cannot be solved analytically for $\kappa \neq 0$ due to the spin–orbit coupling term $\kappa(\kappa-1)/r^2$. In order to obtain an approximate analytical solution of equation (9) with $\kappa \neq 0$, we must make an approximation for the spin–orbit coupling term. We take the following approximation into account for the spin–orbit coupling term,

$$\frac{\kappa(\kappa-1)}{r^2} \approx \frac{4\alpha^2\kappa(\kappa-1)e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}. \tag{10}$$

Such an approximation is a good approximation for small values of the parameter α [36–38]. By employing this approximation into equation (9), we can write down the Schrödiner-like equation (9) for the lower spinor component as

$$\left(-\frac{d^2}{dr^2} - 4(M - E_{n\kappa} + C)e^{-2\alpha r}\left(-\frac{A(A+\alpha)}{(1+e^{-2\alpha r})^2} + \frac{B(B-\alpha)}{(1-e^{-2\alpha r})^2}\right) + \frac{4\kappa(\kappa-1)\alpha^2 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}\right)$$
$$\times G_{n\kappa}(r) = \left(E_{n\kappa}^2 - M^2 - C(M + E_{n\kappa})\right) G_{n\kappa}(r). \tag{11}$$

Defining a new variable $z = -\frac{e^{2\alpha r}}{4}(1 - e^{-2\alpha r})^2$and substituting it into equation (11), we can transform equation (11) into the following form:

$$z(1-z)\frac{d^2 G_{n\kappa}(z)}{dz^2} + \left(\frac{1}{2} - z\right)\frac{dG_{n\kappa}(z)}{dz} + \left[-\varepsilon^2 - \frac{\beta}{z} - \frac{\gamma}{1-z}\right] G_{n\kappa}(z) = 0, \tag{12}$$

where

$$\varepsilon_{n\kappa} = \frac{\sqrt{E_{n\kappa}^2 - M^2 - C(M + E_{n\kappa})}}{2\alpha}, \tag{13a}$$

$$\beta = \frac{(M - E_{n\kappa} + C)B(B-\alpha)}{4\alpha^2} + \frac{1}{4}\kappa(\kappa-1), \tag{13b}$$

$$\gamma = \frac{(M - E_{n\kappa} + C)A(A+\alpha)}{4\alpha^2}. \tag{13c}$$

We analyze the asymptotic behavior of the lower spinor component $G_{n\kappa}(z)$ for the bound states as $r \to 0$ ($z \to 0$). Under this asymptotic condition, we have a solution $G_{n\kappa}(z) = z^{\eta}$ for equation (12), where the parameter η is given by

$$\eta = \frac{1}{4}\left(1 + \sqrt{1 + \frac{4(M - E_{n\kappa} + C)B(B - \alpha)}{\alpha^2} + 4\kappa(\kappa - 1)}\right). \tag{14}$$

Making a transformation of the lower spinor component of the form $G_{n\kappa}(z) = (1 - z)^{\delta} z^{\eta} g_{n\kappa}(z)$, equation (12) becomes

$$(1 - z)z\frac{\mathrm{d}^2 g_{n\kappa}(z)}{\mathrm{d}z^2} + \left[\frac{1}{2} + 2\eta - (1 + 2\delta + 2\eta)z\right]\frac{\mathrm{d}g_{n\kappa}(z)}{\mathrm{d}z} - \left(\varepsilon^2 + 2\delta\eta + \delta^2 + \eta^2 - \frac{\delta^2 - \frac{1}{2}\delta - \gamma}{1 - z}\right) g_{n\kappa}(z) = 0. \tag{15}$$

In order to reduce equation (15) into a hypergeometric equation, we need the following equation to exist,

$$\delta^2 - \tfrac{1}{2}\delta - \gamma = 0. \tag{16}$$

Solving this equation leads us to obtain

$$\delta = \frac{1}{4}\left(1 - \sqrt{1 + \frac{4(M - E_{n\kappa} + C)A(A + \alpha)}{\alpha^2}}\right). \tag{17}$$

If the parameter δ is taken the form given in equation (17), the solution of equation (15) can be expressed in terms of the hypergeometric function, i.e.,

$$g_{n\kappa}(z) = {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a + k)\Gamma(b + k)}{\Gamma(c + k)} \frac{z^k}{k!}, \tag{18}$$

where the parameters a, b and c are given by

$$a = \delta + \eta + \mathrm{i}\varepsilon_{n\kappa}, \tag{19a}$$

$$b = \delta + \eta - \mathrm{i}\varepsilon_{n\kappa}, \tag{19b}$$

$$c = \tfrac{1}{2} + 2\eta. \tag{19c}$$

When either a or b equals to a negative integer $-n$, the hypergeometric function $g_{n\kappa}(z)$ can be reduced to a polynomial of degree n. This shows that the hypergeometric function given in equation (18) can be finite under the following quantum condition:

$$a = -n, \qquad n = 0, 1, 2, 3, \ldots. \tag{20}$$

Substituting this quantum condition (20) into equation (19*a*), we have

$$\varepsilon_{n\kappa}^2 = -(-n - \delta - \eta)^2. \tag{21}$$

Substituting equation (13*a*) into equation (21) and using equations (14) and (17), we obtain the energy eigenvalue equation for the nuclei in the relativistic Pöschl–Teller potential (8) under the exact pseudospin symmetry limit,

$$M^2 - E_{n\kappa}^2 + C(M + E_{n\kappa}) = 4\alpha^2\left(-n - \frac{1}{2} + \frac{1}{4}\sqrt{1 + \frac{4(M - E_{n\kappa} + C)A(A + \alpha)}{\alpha^2}} - \frac{1}{4}\sqrt{1 + 4\kappa(\kappa - 1) + \frac{4(M - E_{n\kappa} + C)B(B - \alpha)}{\alpha^2}}\right)^2, \tag{22}$$

where the quantum number $n = 0, 1, 2, \ldots, < -\delta - \eta$. In the case of the s-wave ($\kappa = 1$), equation (22) becomes

$$M^2 - E_{n\kappa}^2 + C(M + E_{n\kappa}) = 4\alpha^2 \left(-n - \frac{1}{2} + \frac{1}{4}\sqrt{1 + \frac{4(M - E_{n\kappa} + C)A(A + \alpha)}{\alpha^2}} - \frac{1}{4}\sqrt{1 + \frac{4(M - E_{n\kappa} + C)B(B - \alpha)}{\alpha^2}} \right)^2 . \tag{23}$$

This is just expression (31) of [22], which is the energy eigenvalue equation of the Dirac equation with the Pöschl–Teller potential for the case of the spin–orbit quantum number $\kappa = 1$. In terms of the original variable r, the lower component $G_{n\kappa}(r)$ corresponding to the energy level $E_{n\kappa}$ can be expressed as follows:

$$G_{n\kappa}(r) = (1 + \sinh^2 \alpha r)^{\delta} (-\sinh^2 \alpha r)^{\eta} {}_2F_1\left(-n, n + 2(\delta + \eta); \tfrac{1}{2} + 2\eta; -\sinh^2 \alpha r\right). \tag{24}$$

Substituting $G_{n\kappa}(r)$ given in equation (24) into equation (5*b*), we obtain the upper spinor component $F_{n\kappa}(r)$ corresponding to the lower component $G_{n\kappa}(r)$ and energy level $E_{n\kappa}$,

$$\begin{aligned} F_{n\kappa}(r) = \frac{1}{M - E_{n\kappa} + C} \Bigg[& \left(\frac{\alpha\delta \sinh 2\alpha r}{1 + \sinh^2 \alpha r} + \frac{2\alpha\eta \cosh \alpha r}{\sinh \alpha r} - \frac{\kappa}{r} \right) (1 + \sinh^2 \alpha r)^{\delta} (-\sinh^2 \alpha r)^{\eta} \\ & \times {}_2F_1 \left(-n, n + 2(\delta + \eta); \frac{1}{2} + 2\eta; -\sinh^2 \alpha r \right) + \frac{n(n + 2\delta + 2\eta)\alpha \sinh 2\alpha r}{\frac{1}{2} + 2\eta} \\ & \times (1 + \sinh^2 \alpha r)^{\delta} (-\sinh^2 \alpha r)^{\eta} \times {}_2F_1 \Bigg(-n + 1, n + 1 + 2(\delta + \eta); \frac{3}{2} \\ & + 2\eta; -\sinh^2 \alpha r \Bigg) \Bigg]. \end{aligned} \tag{25}$$

From equations (24) and (25), we can see that the lower component $G_{n\kappa}(r)$ and the upper component $F_{n\kappa}(r)$ can satisfy the boundary conditions for the bound states when $\eta > 1$ and $\delta < 0$ and $\eta < -\delta$.

From equation (25) we know that in the limit of pseudospin symmetry there are only bound negative energy states, otherwise the upper spinor component $F_{n\kappa}(r)$ will diverge if $E_{n\kappa} = M$ and $C = 0$. In generally speaking, there are no bound positive energy states under the pseudospin limit condition [8]. The energy level $E_{n\kappa}$ is defined implicitly by energy eigenvalue equation (22) which is a rather complicated transcendental equation. With the help of equation (22), we can determine the energy eigenvalues corresponding to sets of n and κ values by considering that the lower spinor component $G_{n\kappa}(r)$ and the upper spinor component $F_{n\kappa}(r)$ satisfy the bound state restriction conditions, i.e., $\eta > 1$, $\delta < 0$ and $\eta < -\delta$. In order to show the procedure of determining the bound-state energy eigenvalues from equation (22), we take a set of physical parameter values, $\alpha = 0.35$, $A = 1.50$, $B = 1.00$, $M = 5.00$ and $C = -0.35$, to give a numerical example. When $n = 1$ and $\kappa = -1$, equation (22) yields the following values of $E_{1,-1}$: $-4.749\,874$, $4.534\,463$. We choose $E_{1,-1} = -4.749\,874$ as the solution of equation (22), and find that the values of η and δ are $\eta = 3.859\,947$ and $\delta = -7.050\,444$, respectively. If we take $E_{1,-1} = 4.534\,463$ as the solution of equation (22), the values of η and δ are $\eta = 1.096\,028$ and $\delta = -0.596\,650$, which do not satisfy the regularity condition, $\eta < -\delta$. Thus, we can only take the negative energy value $E_{1,-1} = -4.749\,874$ as the solution of equation (22). With the same parameter values of α, A, B, M and C, the numerical solutions of equation (22) for the other values of n and κ are presented in table 1. This table shows the pseudospin partners, i.e., the Dirac eigenstate $1s_{1/2}$ with $n = 1$ and $\kappa = -1$ has a partner eigenstate $0d_{3/2}$ with $n - 1 = 0$ and $\kappa = 2$.

Table 1. The bound-state energy eigenvalues $E_{n\kappa}$ of the pseudospin symmetry Pöschl–Teller potential for several values of n and κ.

$\tilde{l}$	$n, \kappa < 0$	(l, j)	$E_{n,\kappa<0}$	$n-1, \kappa > 0$	$(l+2, j+1)$	$E_{n-1,\kappa>0}$
1	$1, -1$	$1s_{1/2}$	$-4.749\,874$	$0, 2$	$0d_{3/2}$	$-4.749\,874$
2	$1, -2$	$1p_{3/2}$	$-4.779\,318$	$0, 3$	$0f_{5/2}$	$-4.779\,318$
3	$1, -3$	$1d_{5/2}$	$-4.818\,179$	$0, 4$	$0g_{7/2}$	$-4.818\,179$
4	$1, -4$	$1f_{7/2}$	$-4.861\,628$	$0, 5$	$0h_{9/2}$	$-4.861\,628$
1	$2, -1$	$2s_{1/2}$	$-4.923\,104$	$1, 2$	$1d_{3/2}$	$-4.923\,104$
2	$2, -2$	$2p_{3/2}$	$-4.938\,822$	$1, 3$	$1f_{5/2}$	$-4.938\,822$
3	$2, -3$	$2d_{5/2}$	$-4.958\,243$	$1, 4$	$1g_{7/2}$	$-4.958\,243$
4	$2, -4$	$2f_{7/2}$	$-4.977\,479$	$1, 5$	$1h_{9/2}$	$-4.977\,479$

When the parameter α in approximation expression (10) goes to zero, the approximation to the spin–orbit coupling term becomes the following form:

$$\lim_{\alpha\to 0}\left[\frac{4\alpha^2\kappa(\kappa-1)\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2}\right] = \frac{\kappa(\kappa-1)}{r^2}. \tag{26}$$

This shows that the usual spin–orbit coupling term is the limit of the approximation to the spin–orbit coupling term as α goes to zero. Under the limit of α becoming zero, the limits of the energy eigenvalues and eigenfunctions go to

$$\lim_{\alpha\to 0} E_{n\kappa} = (A-B)^2 - M, \tag{27}$$

$$\lim_{\alpha\to 0} G_{n\kappa}(r) = 0, \tag{28}$$

$$\lim_{\alpha\to 0} F_{n\kappa}(r) = 0. \tag{29}$$

From the results given in above three equations, we observe that the lower spinor component $G_{n\kappa}(r)$ and upper spinor component $F_{n\kappa}(r)$ become unbound in the limit of $\alpha \to 0$ and the eigenenergies become constant. In fact, when the limit of α becomes zero, the limit of the difference potential $\Delta(r)$ given in equation (8) goes to

$$\lim_{\alpha\to 0} \Delta(r) = \infty. \tag{30}$$

This shows that the pseudospin symmetry Pöschl–Teller potential does not trap a fermion as α goes to zero. In this limit, the relativistic Pöschl–Teller potential (8) does not possess any bound state under the exact pseudospin symmetry condition.

3. Conclusions

In this paper, we have approximately investigated the bounded solutions of the Dirac equation for the Pöschl–Teller potential with the pseudospin symmetry for the arbitrary spin–orbit quantum number κ. By employing an exponential-type potential to approximate the spin–orbit coupling term $\kappa(\kappa-1)/r^2$, we obtain approximately the energy eigenvalue equation and associated two-component spinors for the pseudospin symmetry Pöschl–Teller potential. Under the exact pseudospin symmetry limit, we recover the energy eigenvalue equation in the Dirac equation with the Pöschl–Teller potential for the case of spin–orbit quantum number $\kappa = 1$.

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